Two-Point Correlations and Critical Line of the Driven Ising Lattice Gas in a High-Temperature Expansion

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Based on a high-temperature expansion, we compute the two-point correlation function and the critical line of an Ising lattice gas driven into a nonequilibrium steady state by a uniform bias E. The lowest nontrivial order already reproduces the key features, i.e., the discontinuity singularity of the structure factor and the (qualitative) E dependence of the critical line. Our approach is easily generalized to other nonequilibrium lattice models and provides a simple *analytic* tool for the study of the high-temperature phase and its boundaries.

I. INTRODUCTION

The study of non-equilibrium steady states (NESS) has attracted vivid interest over the past decade. On the one hand, such studies are application-driven, since NESS determine the physics of a wide range of important problems, including, e.g., granular and traffic flow, surface growth, electromigration, and transport phenomena in biological systems. On the other hand, there is a fundamental interest in creating a theoretical framework for NESS on a par with Gibbs ensemble theory for equilibrium systems. Driven diffusive lattice gases, first introduced by Katz *et al.*[1] and recently reviewed in ref. 2, provide simple testing grounds for the properties of a particular class of NESS. Based mostly on Ising lattice gases with Kawasaki (spin exchange) dynamics, these models are represented by microscopic Master equations which violate the usual detailed balance

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condition by maintaining a net probability current between configurations. A particularly simple member of this class is the uniformly driven lattice gas, in which spin exchanges along a specific lattice axis are biased by a uniform force, remaining energetically controlled only in the transverse subspace. Identifying the particles with (shielded) charges and the force with an electric field, such models provide a starting point for the description of fast ionic conductors[3] or charged droplets in a microemulsion.[4]

One of the more remarkable features of this model, and many of its relatives, is the presence of long-range correlations at all temperatures.[6, 7] Though this behavior can be understood, within the context of a phenomenological (mean) field theory,[7] in terms of the violation of the traditional fluctuation-dissipation theorem,[8] it is clearly important to have an exact microscopic version. Using a venerable tool, the high temperature series expansion, Zhang et al.[6] (ZWLV in the following) investigated G, the two-point correlation function for a system with infinite E. From the master equation and the associated hierarchy, they argued that three-point correlations are negligible and arrived at a closed set of equations for G alone. To zeroth order in β, both the equation and the solution are trivial. To first order, the short-distance behavior of the solution was obtained numerically, by truncating the equations at distances larger than a cutoff value. The results exhibit significant anisotropy and agree quite well with Monte Carlo data at high temperatures, i.e., $T \gtrsim 6$, in units of $T_c(0)$, the critical temperature of the $E = 0$ system. Analytically, the equations were approximated by a Poisson problem with quadrupole symmetry. Its solution captures the behavior of G for large distances, displaying the r^{-2} power law tail, with the angle-dependent amplitude of the quadrupole potential, in agreement with simulations. We should emphasize that the high temperature series has a strong foundation: for $\beta J \equiv 0$, the steady state distribution $P^* \propto 1$ is exactly known for all E,[9] so that we are expanding about a well-defined zeroth order state. In this paper, we generalize the analysis of ZWLV to include finite driving fields and solve the resulting set of equations for G exactly, by computing its Fourier transform, the structure factor S. The latter displays the characteristic discontinuity singularity[1, 7] at the origin, which translates into the r^{-d} decay in real (d-dimensional) space. Thus, recourse to numerical methods is not necessary.

Another key feature of driven lattice gases is the existence of a line of continuous transitions, falling outside the Ising universality class.[10] In particular, it is remarkable that the critical temperature, $T_c(E)$, *increases* with E, even though the drive reduces the effective nearest-neighbor coupling. Apparently, the strong long-range part of the correlations dominates the driven system and permits the onset of order at a higher T_c.[11] To support

this argument, it is desirable to have a simple analytic method to estimate *both* correlations *and* the critical temperature. The high temperature expansion, and specifically our explicit form for the structure factor, serve that purpose. Matching the series expansion of S to the expected critical singularity, we obtain estimates for the critical line, and hence the phase diagram to this (lowest) order in β.

Clearly, quantitative accuracy cannot be expected from a first-order calculation. Nevertheless, it presents one of the *simplest, currently available, tools* for the qualitative prediction of nonuniversal quantities in driven lattice gases. Real-space renormalization group techniques for conserved non-equilibrium spin systems are sorely lacking, and dynamic mean-field theories,[12] while quantitatively more satisfying, are much more labor-intensive. Our method is easily extended to general rate functions, anisotropic interactions and higher dimensions. These results will be published elsewhere.[13]

The paper is organized as follows: we first summarize the model definition and the method of ZWLV, resulting in a closed set of equations for the two-point function G, correct to first order in βJ but for arbitrary drive strength $E > 12J$. These equations are then solved exactly in Fourier space. Key consequences, such as the discontinuity singularity of the structure factor and the critical line $T_c(E)$, are discussed. We conclude with a brief comment on results in higher dimensions.

II. THE MODEL AND THE EQUATIONS FOR ITS TWO-POINT FUNCTION

We first summarize the microscopics of the model. On each site $\vec{r}$ of an (infinite) square lattice in spatial dimension $d = 2$, coupled to a heat bath at inverse temperature β, we define an Ising spin variable $S_{\vec{r}}$ which distinguishes occupied ($S = +1$) from empty ($S = -1$) sites. Interacting via the usual Ising Hamiltonian, $\mathcal{H} = -J \sum S_{\vec{r}} S_{\vec{r}'}$, nearest neighbors can exchange positions, subject to the local energetics and a uniform force E ("electric field") which biases exchanges along a specific lattice axis (aligned with the x-coordinate). To be specific, we choose Metropolis[14] rates, $\min\{1, \exp - \beta[\Delta\mathcal{H} - \sigma E]\}$, where $\sigma = 0, +1, -1$ for jumps transverse, along and against the field. Thus, "infinite" E implies that jumps along the field always take place, provided they are allowed by the excluded volume constraint, while jumps against E are completely suppressed. These rates specify the master equation of the model, for the time-dependent configurational probability P. A hierarchy of equations, connecting different N-point functions, follows as usual. We will be interested in steady-state averages

only, so all time derivatives will be set to zero. Following ZWLV, we expand the rate functions in powers of βJ; as an extension of their work, which focused on $E = \infty$ only, we include the effect of large, but *finite* βE. Thus, in a sense, the case here corresponds to $\beta E \gg \beta J$. In practice, it is sufficient to choose $E > 12J$, so that jumps along the drive occur with unit rate, while those against E are suppressed by a factor of $\exp(-\beta E)$. In a strict high temperature expansion, organized in powers of β, βE would also appear as a small parameter, so that the range $E < 12J$ can also be explored.[13]

Returning to our case, a set of equations for the two-point function, $G(\vec{r}, \vec{r}') \equiv \langle S_{\vec{r}} S_{\vec{r}'} \rangle$, is easily derived from the master equation with the expanded rates. At first order in βJ, one finds that G couples only to three-point functions. Even though, in contrast to the equilibrium Ising model, the latter do not vanish here,[15] they are numerically quite small and are neglected. A closed set of equations for G emerges. By translational invariance, G depends only on (x, y), the separation between the two points $\vec{r}$ and $\vec{r}'$, and it is even in both variables.

The resulting equations depend on two parameters,

$$K \equiv \beta J \quad \text{and} \quad \varepsilon \equiv \exp(-\beta E)$$

Up to and including only first order terms in K, we obtain a set of linear equations for G,

$$0 = 2[G(1, 1) + G(1, -1) - 2G(1, 0)] + (1 + \varepsilon)$$
$$\times [G(2, 0) - G(1, 0)] + 2K(2 + \varepsilon)$$

$$0 = 2[G(0, 2) - G(0, 1)] + (1 + \varepsilon)$$
$$\times [G(1, 1) + G(-1, 1) - 2G(0, 1)] + 2K(1 + 2\varepsilon)$$

$$0 = 2[G(1, 2) + G(1, 0) - 2G(1, 1)] + (1 + \varepsilon)$$
$$\times [G(2, 1) + G(0, 1) - 2G(1, 1)] - 2K(1 + \varepsilon) \tag{1}$$

$$0 = 2[G(2, 1) + G(2, -1) - 2G(2, 0)] + (1 + \varepsilon)$$
$$\times [G(3, 0) + G(1, 0) - 2G(2, 0)] - 2\varepsilon K$$

$$0 = 2[G(0, 3) + G(0, 1) - 2G(0, 2)] + (1 + \varepsilon)$$
$$\times [G(1, 2) + G(-1, 2) - 2G(0, 2)] - 2K$$

and, for all other non-zero x, y:

$$0 = 2[G(x, y+1) + G(x, y-1) - 2G(x, y)] + (1 + \varepsilon)$$
$$\times [G(x+1, y) + G(x-1, y) - 2G(x, y)] \qquad (2)$$

Here, we have written the equations in such a form that the left hand side would simply be $\partial_t G(x, y)$. The ZWLV equations can be retrieved by setting $\varepsilon = 0$. The solution to these equations is most easily found in Fourier space. In other words, we consider the structure factor:

$$S(k, p) = \sum_{x, y}^{\infty} G(x, y) \exp[-i(kx + py)]$$

which is real, since G is even. Some details on Fourier transforms are given in Appendix A.

In the absence of J, the solution is trivial:

$$G(0, 0) = 1$$
$$G(x, y) = 0, \qquad x, y \neq 0$$

so that

$$S(k, p) = 1 \equiv \bar{S}$$

reflecting, of course, that $P^* \propto 1$ in this case.[9] So, the information about interactions is carried by $\tilde{S}$, defined by:

$$S = \bar{S} + \tilde{S}$$

The correlations appearing in (1), being proportional to K, are really the transforms of $\tilde{S}$:

$$G(x, y) = \int \tilde{S}(k, p) \exp i(kx + py), \qquad (x, y) \neq (0, 0) \qquad (3)$$

For convenience, we have introduced the notation

$$\int \equiv \frac{1}{(2\pi)^2} \int_{-\pi}^{+\pi} dk \int_{-\pi}^{+\pi} dp$$

and will denote the anisotropic lattice Laplacian by

$$\Delta(k, p) \equiv 2(1 + \varepsilon)(1 - \cos k) + 4(1 - \cos p)$$

Inserting (3) into (1, 2) and using the fact that $\tilde{S}$ is real, we obtain

$$\int \tilde{S}\Delta \exp(ik) + (1 + \varepsilon) \int \tilde{S}(1 - \cos k) = +2(2 + \varepsilon) K$$

$$\int \tilde{S}\Delta \exp(ip) + 2 \int \tilde{S}(1 - \cos p) = +2(1 + 2\varepsilon) K$$

$$\int \tilde{S}\Delta \exp[i(k + p)] = -2(1 + \varepsilon) K \qquad (4)$$

$$\int \tilde{S}\Delta \exp(2ik) = -2\varepsilon K$$

$$\int \tilde{S}\Delta \exp(2ip) = -2K$$

with similar equations for negative values of x or y and

$$\int \tilde{S}\Delta \exp[i(kx + py)] = 0 \qquad \text{for} \quad |x| + |y| > 2 \qquad (5)$$

Next, we seek to invoke the completeness relation (see the Appendix) in order to project out an equation for $\tilde{S}$. While the two additional terms on the left hand side of (4) might appear to spoil this approach, we can just treat them, for the time being, as unknown ε-dependent coefficients,

$$I_1 \equiv \int \tilde{S}(1 - \cos k)$$
$$(6)$$
$$I_2 \equiv \int \tilde{S}(1 - \cos p)$$

and move them to the right hand side. Finally, for completeness, we need an additional equation for $x = y = 0$, namely:

$$\int \tilde{S}\Delta = \int \tilde{S}[2(1 + \varepsilon)(1 - \cos k) + 4(1 - \cos p)] = 2(1 + \varepsilon) I_1 + 4I_2 \qquad (7)$$

Now that we have equations for *all* integer values of x, y, we can use $\sum_{x,y} \exp[i(kx + py)] = (2\pi)^2 \delta(k) \delta(p)$. The result is

$$\tilde{S}(k, p) = L(k, p)/\Delta(k, p) \qquad (8)$$

where L is the sum of terms on the right hand side, i.e.,

$$L(k, p) = 2I_1(1 + \varepsilon)(1 - \cos k) + 4I_2(1 - \cos p)$$
$$+ 4K[\varepsilon(1 - \cos k) + (1 - \cos p)][1 + 2 \cos k + 2 \cos p] \qquad (9)$$

Of course, (8) is still an implicit equation for the structure factor, due to the appearance of I_1 and I_2 on the right. To find the explicit solution, we must determine the I's. Since we have two unknowns, we need two linearly independent equations. The first of these follows from (6): Inserting our result for $L(k, p)$, Eq. (9) into the first equation of (6), we obtain

$$0 = -I_1 + \int \frac{L(k, p)}{\Delta(k, p)} (1 - \cos k) = M_{1, j}I_j + KN_1$$

where

$$M_{1, 1} = -1 + 2(1 + \varepsilon) \int (1 - \cos k)^2 / \Delta(k, p)$$

$$M_{1, 2} = 4 \int (1 - \cos k)(1 - \cos p) / \Delta(k, p)$$

and

$$N_1 = \int (1 - \cos k)[4(1 - \cos p) + 4\varepsilon(1 - \cos k)]$$
$$\times [2 \cos k + 2 \cos p + 1] / \Delta(k, p)$$

Due to a remarkable symmetry under (k, p) exchange,[13] the second equation in (6) is *not* linearly independent from the first. Instead, an additional equation is provided by the value of G at the origin, i.e., $1 = G(0, 0) = \int S(k, p)$ which leads to

$$0 = \int \tilde{S}(k, p) = \int \frac{L(k, p)}{\Delta(k, p)} = M_{2, j}I_j + KN_2$$

with

$$M_{2, 1} = 2(1 + \varepsilon) \int (1 - \cos k) / \Delta(k, p)$$

$$M_{2, 2} = 4 \int (1 - \cos p) / \Delta(k, p)$$

and

$$N_2 = \int \left[4(1 - \cos p) + 4\varepsilon(1 - \cos k)\right]\left[2 \cos k + 2 \cos p + 1\right]/\Delta(k, p)$$

Note that the singularity of $1/\Delta(k, p)$ at the origin is cancelled by zeros in the numerators, so that all of these integrals are perfectly finite. The explicit solution now follows as

$$I_m = -KM_{mn}^{-1} N_n \tag{10}$$

Together with (9), this determines the full structure factor, displaying the expected proportionality $\tilde{S}(k, p) \propto K$.

For later reference, let us briefly consider the equilibrium analog to our results so far. To first non-trivial order in K, the two-point correlation of the Ising model is given by the well-known form[5]

$$G_{eq}(0, 0) = 1$$

$$G_{eq}(1, 0) = G_{eq}(0, 1) = K$$

$$G_{eq}(x, y) = 0 \qquad \text{for all other} \quad x, y$$

This results in a structure factor $S_{eq}(k, p) = 1 + 2K(\cos k + \cos p)$ which is of course isotropic.

III. DISCONTINUITY SINGULARITY AND CRITICALITY

In this section, we discuss two of the most interesting consequences of the full solution,

$$S = \bar{S} + \frac{L}{\Delta} + O(K^2) \tag{11}$$

First, we focus on the celebrated discontinuity of the structure factor near the origin.[1,7] This anomaly is a direct consequence of FDT violation and is therefore expected to increase in magnitude with the strength of the drive. For small k, p we find

$$\Delta = (1 + \varepsilon) k^2 + 2p^2 + O(k^4, k^2 p^2, p^4)$$

$$L = \left[(1 + \varepsilon) I_1 + 10\varepsilon K\right] k^2 + 2\left[I_2 + 5K\right] p^2 + O(k^4, k^2 p^2, p^4)$$

Thus, the discontinuity can be measured by

$$\lim_{p \to 0} S(0, p) - \lim_{k \to 0} S(k, 0) = I_2 - I_1 + 5K \frac{1 - \varepsilon}{1 + \varepsilon}$$

Of course, this is proportional to K. The dependence on E is captured by the parameter ε, which also enters into the expressions for the integrals I_1 and I_2 (see the Appendix for a discussion and some characteristic values). Consistent with our expectation, we observe that the discontinuity increases monotonically with E, from a limiting value of 0 for $12K < \beta E \ll 1$ to 5.48 for infinite drive.

Originally obtained from field-theoretic considerations, the form which best displays this discontinuity is,[2, 10]

$$S(k, p) = \frac{n_{\|} k^2 + n_{\perp} p^2}{\tau_{\|} k^2 + \tau_{\perp} p^2 + O(k^4, k^2 p^2, p^4)} \tag{12}$$

near the origin. Here, $n_{\|}$ and $n_{\perp}$ measure the strength of thermal noise in the parallel and transverse directions, respectively, while $\tau_{\|}$ and $\tau_{\perp}$ are the anisotropic diffusion coefficients. Under equilibrium conditions, the FDT enforces the equality

$$\frac{n_{\|}}{n_{\perp}} = \frac{\tau_{\|}}{\tau_{\perp}}$$

so that the familiar Ornstein–Zernike form re-emerges. By contrast, we conclude that

$$\frac{n_{\perp}}{\tau_{\perp}} - \frac{n_{\|}}{\tau_{\|}} = I_2 - I_1 + 5K \frac{1 - \varepsilon}{1 + \varepsilon} \neq 0$$

in the driven case.

Next, we turn to an estimate for the critical temperature. If we had the exact $S(k, p)$, we could identify T_c by its divergence at some point. For the usual system in equilibrium, we would look for the divergence of $S(0, 0)$, which is the susceptibility (τ^{-1}). However, with conserved dynamics at half-filling, this quantity is fixed at zero, so that it is necessary to consider $\lim_{k \to 0} S(k)$. In our case, as we just pointed out, this limit depends on the direction along which we approach the origin. Thus, we appeal first to the phenomenology, i.e., only $\lim_{p \to 0} S(0, p)$ diverges as $T \to T_c$. In terms of (12), only $n_{\perp}/\tau_{\perp} \to \infty$. In terms of co-operative behavior, this corresponds to an instability against ordering into strips *parallel* to the drive only.

Now, in a high temperature series, where only a finite number of terms can be computed, every partial sum is finite. Instead, the radius of convergence must be estimated. However, here we have only one non-trivial term! To make any estimate, we turn to a search for the *zero* of S^{-1} instead. For the equilibrium case, $S_{eq}^{-1}(k, p) = 1 - 2K(\cos k + \cos p) + O(K^2)$, so that this procedure leads to $T_c = 4J/k_B$. Of course, this is the same result, had we expanded the exact equation for β_c ($e^{-4\beta_c J} + 2e^{-2\beta_c J} = 1$) in powers of $\beta_c J$ and kept only the first non-trivial term. Remarkably, this is also the mean field critical temperature.

Turning to the driven case, we consider an expansion of S^{-1} (given that $\bar{S} = 1$):

$$S^{-1} = 1 - \frac{L}{\Delta} + O(K^2) \tag{13}$$

and seek the zero of $\lim_{p \to 0} S^{-1}(0, p)$. The result is

$$T_c(E) = (J/k_B)[5 + I_2/K] \tag{14}$$

Recall that I_2 is proportional to K, so that their ratio is a β-independent number (Table 1).

Before discussing the implications of this result, let us identify (up to an overall constant) the various parameters in Eq. (12) using this approach:

$$\tau_{||} = (1 + \varepsilon)(1 - I_1) - 10K\varepsilon$$

$$\tau_{\perp} = 2(1 - I_2 - 5K)$$

$$n_{||} = 1 + \varepsilon$$

$$n_{\perp} = 2$$

Referring to the discussion of I_1 and I_2 given in the Appendix, we see that $\tau_{||} > \tau_{\perp}$ for all $E > 0$. This inequality confirms our choice to identify the critical temperature by the vanishing of $\tau_{\perp}$.

Returning to (14), let us discuss critical temperatures in units of J/k_B. First, note that I_2 is negative and monotonically decreasing in $\varepsilon = \exp(-\beta E)$ (cf. Appendix). Therefore, $T_c(E)$ increases with E, taking its maximum at infinite E where $T_c(\infty) = 4.640$. For the smallest βE, defined through the inequality $12K < \beta E \ll 1$, $T_c(E)$ approaches its lowest value of 4.0. The agreement with the equilibrium result, $T_c(0) = 4$, is an artifact of the *lowest* order of the expansion only.[13] Clearly, it is gratifying that even the lowest nontrivial order of the high-temperature expansion generates a $T_c(E)$ which *increases* with E, in *qualitative* agreement with simulation data.

Finally, let us consider the *quantitative* implications of our results by focusing on the ratio $T_c(E)/T_c(0)$. For $E = \infty$, our approach yields the value 1.16, while MC simulations result in 1.40.[16] Thus, the series underestimates this ratio, which can be understood as follows. The high temperature series is known to overestimate critical temperatures, by underestimating fluctuations. However, it has been argued[11] that the external drive tends to suppress fluctuations, so that we may expect $T_c(\infty)$ to be *less* sensitive to the numerical errors introduced by the high temperature series than its equilibrium counterpart. In this sense, the series expansion should be "better" for a driven system. Indeed, we compare the series result of 4.64 to the simulation value 3.18, finding a discrepancy of 46%. In contrast, for the equilibrium case, we have 4.00 and 2.27 respectively, showing a much higher discrepancy of 76%.

Finally, we could hope for better agreement of series and the exact T_c as we move into higher spatial dimensions, where fluctuations become less important. The results are certainly encouraging in $d = 3$. In the series approach, we obtain $T_c(0) = 6$ and $T_c(\infty) = 6.34$, signalling an increase of 6% due to the drive. This is in remarkably good agreement with the MC data, which show a 7% increase.[17] Thus, even a low-order calculation can produce some quantitatively reliable results.

IV. CONCLUSIONS

Within a high temperature series, we have derived the equations for the two-point correlations of the uniformly driven lattice gas to lowest non-trivial order in βJ, but finite βE. The exact solution of these equations provides a successful qualitative description of two central features of our model, namely, the discontinuity singularity of the structure factor at the origin, associated with power-law correlations in the disordered phase, and the anisotropy in the parallel and transverse diffusion coefficients which controls the onset of criticality. Specifically, we observe that the magnitude of the structure factor discontinuity increases with E, as a measure of how seriously the FDT is violated in the driven system. We demonstrate explicitly that criticality is marked by the vanishing of the *transverse* diffusion coefficient, resulting in an estimate for $T_c(E)$ which increases with E, consistent with MC data. On the quantitative side, we argue that fluctuations, largely neglected in a series expansion such as ours, tend to increase the ratio $T_c(E)/T_c(0)$. In higher spatial dimensions, where fluctuations are less relevant, agreement of series and MC data improves, borne out by our results in $d = 3$.

Quantitative comparisons aside, the approach presented here provides a convenient analytic complement to MC simulations, since it gives direct

microscopic information about effective coarse-grained coupling constants such as $\tau_\perp$ and $\tau_\parallel$ which appear in field theories. It is computationally simpler than dynamic mean-field theory and easily generalized[13] to higher dimensions or other driven lattice models. Thus, it can help to predict qualitative phase diagrams and formulate effective field theories for a wide range of non-equilibrium steady states.

APPENDIX

Here, we give a few details of our calculations. First, we briefly review the conventions of our Fourier transforms and then turn to the evaluation of the integrals contributing to (10).

Since our lattice consists of discrete points, we will let (x, y) be all pairs of integers. For our case, it is most convenient to define the functions $U_{x, y}(k, p)$:

$$U_{x, y}(k, p) \equiv \frac{1}{2\pi} \exp i(kx + py)$$

with continuous $k, p \in [-\pi, \pi]$. The $U_{x, y}$ form a complete orthonormal set:

$$\sum_{x, y} U_{x, y}(k, p) \, U_{x, y}(k', p') = \delta(k - k') \, \delta(p - p')$$

$$\int dk \, dp \, U_{x, y}(k, p) \, U_{x', y'}(k, p) = \delta_{xx'} \, \delta_{yy'}$$

The Fourier transform is defined in the usual way,

$$G(x, y) = \int S(k, p) \exp i(kx + py)$$

with inverse

$$S(k, p) = \sum_{x, y} G(x, y) \exp[-i(kx + py)]$$

Note that $S(k, p)$ is real, since $G(x, y)$ is even in both of its arguments.

Next, we turn to the integrals contributing to (10). In order to exhibit their properties succinctly, it will be helpful to write the anisotropic Laplacian in the form

$$\Delta(k, p) \equiv A(1 - \cos k) + B(1 - \cos p)$$

where the values of interest, $A = 2(1 + \varepsilon)$ and $B = 4$ will be inserted at the end. It is then easily seen from Eqs. (6) and (9), that all integrals are of the general form

$$R_{ij}(A, B) \equiv \int \frac{(1 - \cos k)^i (1 - \cos p)^j}{\Delta(k, p)}$$

Clearly, $R_{ij}(A, B) = R_{ji}(B, A)$. Specifically, we need all pairs (i, j) with $i, j = 0, 1, 2, 3$ except $(0, 0)$. The calculations are simplified by a series of identities, namely,

$$1 = \int \frac{\Delta(k, p)}{\Delta(k, p)} = AR_{10} + BR_{01}$$

$$1 = \int \frac{(1 - \cos k) \Delta(k, p)}{\Delta(k, p)} = AR_{20} + BR_{11}$$

$$1 = \int \frac{(1 - \cos p) \Delta(k, p)}{\Delta(k, p)} = AR_{11} + BR_{02}$$

$$\frac{3}{2} = \int \frac{(1 - \cos k)^2 \Delta(k, p)}{\Delta(k, p)} = AR_{30} + BR_{21}$$

$$1 = \int \frac{(1 - \cos k)(1 - \cos p) \Delta(k, p)}{\Delta(k, p)} = AR_{21} + BR_{12}$$

$$\frac{3}{2} = \int \frac{(1 - \cos p)^2 \Delta(k, p)}{\Delta(k, p)} = AR_{12} + BR_{03}$$

It is thus sufficient to compute R_{i0} only. After performing the elementary integral over p, the substitution $1 - \cos k \equiv 2t$ generates, up to prefactors, the integral representation of Gauss' hypergeometric function.[18] Defining $z \equiv A/B$, we obtain

$$R_{10} = \frac{2}{\pi B \sqrt{z}} F\left(\frac{1}{2}, 1; \frac{3}{2}; -z\right)$$

$$R_{20} = \frac{8}{3\pi B \sqrt{z}} F\left(\frac{1}{2}, 2; \frac{5}{2}; -z\right)$$

$$R_{30} = \frac{64}{15\pi B \sqrt{z}} F\left(\frac{1}{2}, 3; \frac{7}{2}; -z\right)$$

Table 1. Characteristic Values for I_1 and I_2 in
Units of K, for Different Field Strengths

ε	0	0.5	1
I_1/K	-0.844	-0.931	-1
I_2/K	-0.360	-0.764	-1

We note in passing that all of these integrals can be expressed through elementary functions, by reducing the hypergeometric functions down to $F(\frac{1}{2}, 1; \frac{3}{2}, -z) = z^{-1/2} \arctan \sqrt{z}$. It is now straightforward, if somewhat tedious, to compute I_1 and I_2, and hence $\tau_{\parallel}$ and $\tau_{\perp}$, as functions of $z = (1 + \varepsilon)/2$, in the region of interest $0 \leqslant \varepsilon \equiv \exp(-\beta E) < 1$. Both I's are negative and decrease monotonically. Since the explicit forms are not particularly illuminating, we quote a few representative values in Table 1.

ACKNOWLEDGMENTS

REFERENCES

1. S. Katz, J. L. Lebowitz, and H. Spohn, *Phys. Rev. B* **28**:1655 (1983) and *J. Stat. Phys.* **34**:497 (1984).
2. B. Schmittmann and R. K. P. Zia, in *Phase Transitions and Critical Phenomena*, Vol. 17, C. Domb and J. L. Lebowitz, eds. (Academic Press, London, 1995).
3. S. Chandra, *Superionic Solids. Principles and Applications* (North Holland, Amsterdam, 1981).
4. G. Gompper and M. Schick, in *Phase Transitions and Critical Phenomena*, Vol. 16, C. Domb and J. L. Lebowitz, eds. (Academic Press, London, 1994); L. Auvray, in *Micelles, Membranes and Monolayers*, W. M. Gelbart, A. Ben-Shaul, and D. Roux, eds. (Springer, Berlin, 1994).
5. C. Domb, in *Phase Transitions and Critical Phenomena*, Vol. 3, C. Domb and M. Green, eds. (Academic Press, London, 1974).
6. M. Q. Zhang, J.-S. Wang, J. L. Lebowitz, and J. L. Valles, *J. Stat. Phys.* **52**:1461 (1988). See also P. L. Garrido, J. L. Lebowitz, C. Maes, and H. Spohn, *Phys. Rev. A* **42**:1954 (1990).
7. R. K. P. Zia, K. Hwang, B. Schmittmann, and K.-t. Leung, *Physica A* **194**:183 (1993).
8. R. Kubo, *Rep. Progr. Phys.* **29**:25 (1966).
9. F. Spitzer, *Adv. Math.* **5**:246 (1970).

10. H. K. Janssen and B. Schmittmann, *Z. Phys. B* **64**:503 (1986); K.-t. Leung and J. L. Cardy, *J. Stat. Phys.* **44**:567 (1986) and **45**:1087 (1986).

11. C. C. Hill, R. K. P. Zia, and B. Schmittmann, *Phys. Rev. Lett.* **77**:514 (1996).

12. R. Dickman, *Phys. Rev. B* **38**:2588 (1988), and N. C. Pesheva, Y. Shnidman, and R. K. P. Zia, *J. Stat. Phys.* **70**:737 (1993).

13. L. B. Shaw, B. Schmittmann, and R. K. P. Zia, to be published.

14. N. Metropolis, A. W. Rosenbluth, M. M. Rosenbluth, A. H. Teller, and E. Teller, *J. Chem. Phys.* **21**:1087 (1953).

15. K. Hwang, B. Schmittmann, and R. K. P. Zia, *Phys. Rev. Lett.* **67**:326 (1991) and *Phys. Rev. E* **48**:800 (1993).

16. K.-t. Leung, *Phys. Rev. Lett.* **66**:453 (1991) and *Int. J. Mod. Phys. C* **3**:367 (1992); J.-S. Wang, *J. Stat. Phys.* **82**:1409 (1996).

17. J. Marro, J. L. Lebowitz, H. Spohn, and M. H. Kalos, *J. Stat. Phys.* **38**:725 (1985).

18. M. Abramovitz and I. A. Stegun, *Handbook of Mathematical Functions* (Washington, 1965).